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Research Article

Optimality and Duality in Nonsmooth Multiobjective Optimization Involving V-Type I Invex Functions

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A new class of generalized V-type I invex functions is introduced for nonsmooth multiobjective programming problem. Based upon these generalized invex functions, we establish sufficient optimality conditions for a feasible point to be an efficient or a weakly efficient solution. Weak, strong, and strict converse duality theorems are proved for Mond-Weir type dual program in order to relate the weakly efficient solutions of primal and dual programs.

1. Introduction

There is a vital role of convexity in many aspects of mathematical programming including optimality conditions, duality theorems, and alternative theorems, but, due to insufficiency of convexity notion in many mathematical models used in decision science, economics, engineering, and so forth, there has been an increasing interest in relaxing convexity assumptions in connection with sufficiency and duality theorems. One of the most lively generalizations of convexity is owed to Hanson [1], which was named as invexity by Craven [2]. Later, Hanson and Mond [3] defined two new classes of functions, called type I and type II functions, which have been further generalized by many researchers and applied to nonlinear programming problems in different settings. This concept was further generalized to pseudo-type I and quasi-type I functions by Rueda and Hanson [4] and to pseudo-quasi-type I, quasi-pseudo-type I, and strictly quasi-pseudo-type I functions by Kaul et al. [5].

Since many practical problems encountered in economics, engineering design, and management science, and so forth can be described only by nonsmooth functions, consequently, the theory of nonsmooth optimization using locally Lipschitz functions was put forward by Clarke in 1980's (see [6]). He extended the properties of convex functions to the case of locally Lipschitz functions by suitably defining a generalized derivative and a subdifferential. Later on, the notion of invexity was extended to locally Lipschitz functions by Craven [7], by replacing the derivative with Clarke's generalized gradient. Reiland [8] pointed out that under the invexity assumption, the Kuhn-Tucker conditions also assure the optimality in nondifferentiable programming involving locally Lipschitz functions. Recent development of optimality conditions and duality relations for nonsmooth multiobjective programming problems involving locally Lipschitz functions can be seen in [9–17].

In order to resolve the difficulty of demanding same function η for objective and constraint functions in problems dealing with invexity, Jeyakumar and Mond [18] introduced the concept of V-invexity and its generalization for differentiable multiobjective programming problems. However, the extension of their studies to nonsmooth case was discussed by Egudo and Hanson [9]. Further development in this direction can be found in [14, 17]. Zhao [19] established optimality conditions and duality results in nonsmooth scalar programming assuming Clarke's generalized subgradients under type I functions [6]. Kuk and Tanino [15] obtained Karush-Kuhn-Tucker type necessary and sufficient optimality conditions and duality theorems for nonsmooth multiobjective programming problems involving generalized type I vector-valued functions.

In this paper, we are motivated by Kuk and Tanino [15] to introduce generalized type I invex functions, called generalized V-type I invex functions, an extension of V-type I functions introduced by Hanson et al. [20] to nonsmooth cases. By utilizing these new concepts, we obtain Karush-Kuhn-Tucker type sufficient optimality conditions and Mond-Weir type duality relations for nonsmooth multiobjective programming problems. Our results generalize a variety of previously known results in this area.

2. Notations and Preliminaries

Throughout the paper, we use the following conventions of vectors in $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$, $x \geqq y \Leftrightarrow x_i \geqq y_i$, $i = 1, 2, \ldots, n$, $x \geq y \Leftrightarrow x \geqq y$, $x \neq y$, and $x > y \Leftrightarrow x_i > y_i$, $i = 1, 2, \ldots, n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *locally Lipschitz at a point* $\overline{x} \in \mathbb{R}^n$ if there exist scalars $\zeta > 0$ and $\epsilon > 0$ such that

$$\left| f\left(x^1\right) - f\left(x^2\right) \right| \leqq \zeta \left\| x^1 - x^2 \right\|, \quad \forall x^1, x^2 \in \overline{x} + \epsilon B, \tag{2.1}$$

where $\overline{x} + \epsilon B$ is the open ball of radius ϵ around $\overline{x}$ and $\|\cdot\|$ is any norm in $\mathbb{R}^n$.

The *Clarke generalized directional derivative* [6] of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ at $\overline{x}$ in the direction $v \in \mathbb{R}^n$, denoted by $f^\circ(\overline{x}; v)$, is defined as

$$f^\circ(\overline{x}; v) = \limsup_{\substack{y \to \overline{x} \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}, \tag{2.2}$$

where y is a vector in $\mathbb{R}^n$.

The *Clarke generalized gradient* [6] of $f : \mathbb{R}^n \to \mathbb{R}$ at $\overline{x}$, denoted by $\partial^c f(\overline{x})$, is defined as

$$\partial^c f(\overline{x}) = \left\{ \xi \in \mathbb{R}^n : f^\circ(\overline{x}; v) \geqq \xi^{\mathrm{T}} v,\ \forall v \in \mathbb{R}^n \right\}. \tag{2.3}$$

It follows that for any $v \in R^n$, $f^\circ(\overline{x}; v) = \max\{\xi^T v : \xi \in \partial^c f(\overline{x})\}$.
We consider the following nonlinear multiobjective programming problem:

$$\begin{aligned} &\text{Minimize} \quad f(x) = (f_1(x), f_2(x), \ldots, f_k(x)), \\ &\text{subject to} \quad x \in S = \{x \in X : g(x) \leqq 0\}, \end{aligned} \tag{MP}$$

where $X \subseteq \mathbb{R}^n$ is an open set and the functions $f = (f_1, f_2, \ldots, f_k) : X \to \mathbb{R}^k$ and $g = (g_1, g_2, \ldots, g_m) : X \to \mathbb{R}^m$ are locally Lipschitz on X.

Since the objectives in multiobjective programming problems generally conflict with one another, an optimal solution is chosen from the set of efficient (weakly efficient) solutions in the following sense (see [21]).

Definition 2.1. A point $\overline{x} \in S$ is said to be an *efficient solution* of (MP) if there exists no $x \in S$ such that $f(x) \leq f(\overline{x})$.

Definition 2.2. A point $\overline{x} \in S$ is said to be a *weakly efficient solution* of (MP) if there exists no $x \in S$ such that $f(x) < f(\overline{x})$.

Let $K = \{1, 2, \ldots, k\}$, and let $M = \{1, 2, \ldots, m\}$ be any index set. For $\overline{x} \in S$, $J(\overline{x}) = \{j \in M : g_j(\overline{x}) = 0\}$ and g_J denotes the vector of active constraints at $\overline{x}$.

We define the following generalized V-type I invex functions. Let f and g be locally Lipschitz functions at a given point $u \in X$.

Definition 2.3. The pair (f, g) is said to be V-type I invex at $u \in X$ if for each $x \in S$ and for any $\xi_i \in \partial^c f_i(u), \zeta_j \in \partial^c g_j(u)$, there exist vectors α_i and β_j, where $\alpha_i, \beta_j : X \times X \to \mathbb{R}_+ \setminus \{\mathbf{0}\}$, and a function $\eta : S \times X \to \mathbb{R}^n$ such that for all $i \in K, j \in M$

$$\begin{gathered} f_i(x) - f_i(u) \geqq \alpha_i(x, u) \xi_i \eta(x, u), \\ -g_j(u) \geqq \beta_j(x, u) \zeta_j \eta(x, u). \end{gathered} \tag{2.4}$$

Remark 2.4. If $\alpha_i(x, u) = \beta_j(x, u) = 1$, for $i \in K$, $j \in M$, we obtain the definition of type I function given by Kuk and Tanino [15].

Definition 2.5. The pair (f, g) is said to be V-pseudo-quasi-type I invex at $u \in X$ if for each $x \in S$ and for any $\xi_i \in \partial^c f_i(u)$, $\zeta_j \in \partial^c g_j(u)$, there exist vectors $\tilde{\alpha}_i$ and $\tilde{\beta}_j$, where $\tilde{\alpha}_i, \tilde{\beta}_j : X \times X \to \mathbb{R}_+ \setminus \{\mathbf{0}\}$, and a function $\eta : S \times X \to \mathbb{R}^n$ such that

$$\begin{gathered}\sum_{i=1}^{k} \tilde{\alpha}_i f_i(x) < \sum_{i=1}^{k} \tilde{\alpha}_i f_i(u) \Longrightarrow \sum_{i=1}^{k} \xi_i \eta(x, u) < 0, \\ -\sum_{j=1}^{m} \tilde{\beta}_j g_j(u) \leqq 0 \Longrightarrow \sum_{j=1}^{m} \zeta_j \eta(x, u) \leqq 0.\end{gathered} \tag{2.5}$$

If in the above definition first inequality is satisfied as

$$\sum_{i=1}^{k} \tilde{\alpha}_i f_i(x) \leqq \sum_{i=1}^{k} \tilde{\alpha}_i f_i(u) \Longrightarrow \sum_{i=1}^{k} \xi_i \eta(x, u) < 0, \tag{2.6}$$

then we say that (f, g) is V-strictly pseudo-quasi-type I invex at u.

Definition 2.6. The pair (f, g) is said to be V-quasi-pseudo-type I invex at $u \in X$ if for each $x \in S$ and for any $\xi_i \in \partial^c f_i(u)$, $\zeta_j \in \partial^c g_j(u)$, there exist vectors $\hat{\alpha}_i$ and $\hat{\beta}_j$, where $\hat{\alpha}_i, \hat{\beta}_j : X \times X \to \mathbb{R}_+ \setminus \{0\}$, and a function $\eta : S \times X \to \mathbb{R}^n$ such that

$$\begin{gathered}\sum_{i=1}^{k} \xi_i \eta(x, u) > 0 \Longrightarrow \sum_{i=1}^{k} \hat{\alpha}_i f_i(x) > \sum_{i=1}^{k} \hat{\alpha}_i f_i(u), \\ -\sum_{j=1}^{m} \hat{\beta}_j g_j(u) < 0 \Longrightarrow \sum_{j=1}^{m} \zeta_j \eta(x, u) < 0.\end{gathered} \tag{2.7}$$

If in the above definition second inequality is satisfied as

$$-\sum_{j=1}^{m} \hat{\beta}_j g_j(u) \leqq 0 \Longrightarrow \sum_{j=1}^{m} \zeta_j \eta(x, u) < 0, \tag{2.8}$$

then we say that (f, g) is V-quasistrictly pseudo-type I invex at u.

We will need the following result.

Theorem 2.7 (see [21, page 45]). *Let the functions $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, 2, \cdots, k)$ be locally Lipschitzian at a point $x^* \in \mathbb{R}^n$. Then, for weights $w_i \in \mathbb{R}$, one has*

$$\partial^c \left(\sum_{i=1}^{k} w_i f_i \right)(x^*) \subset \sum_{i=1}^{k} w_i \partial^c f_i(x^*). \tag{2.9}$$

3. Karush-Kuhn-Tucker Type Sufficient Optimality Conditions

In this section, we derive some sufficient optimality conditions for a feasible solution to be an efficient or a weakly efficient solution for (MP). Throughout this section, and in Section 4, $f^{\overline{\lambda}}$ denotes the vector $(\overline{\lambda}_1 f_1, \overline{\lambda}_2 f_2, \ldots, \overline{\lambda}_k f_k)$ and $g_J^{\overline{\mu}}$ denotes the vector whose components are $\overline{\mu}_j g_j$, $j \in J(\overline{x})$.

Theorem 3.1. *Suppose that there exist a feasible solution $\overline{x}$ for* (MP) *and scalars $\overline{\lambda}_i > 0, i \in K, \overline{\mu}_j \geqq 0, j \in J(\overline{x})$ such that*

(i) $0 \in \sum_{i=1}^{k} \overline{\lambda}_i \partial^c f_i(\overline{x}) + \sum_{j \in J(\overline{x})} \overline{\mu}_j \partial^c g_j(\overline{x})$,

(ii) (f, g_J) *is V-type I invex at $\overline{x}$.*

Then, $\overline{x}$ is an efficient solution for (MP).

Proof. Hypothesis (i) implies that there exist $\xi_i \in \partial^c f_i(\overline{x}), i \in K$ and $\zeta_j \in \partial^c g_j(\overline{x}), j \in J(\overline{x})$ satisfying

$$0 = \sum_{i=1}^{k} \overline{\lambda}_i \xi_i + \sum_{j \in J(\overline{x})} \overline{\mu}_j \zeta_j. \tag{3.1}$$

Since (f, g_J) is V-type I invex at $\overline{x}$, we have for all $x \in S$

$$\begin{gathered} f_i(x) - f_i(\overline{x}) \geqq \alpha_i(x, \overline{x}) \xi_i \eta(x, \overline{x}), \quad \text{for any } \xi_i \in \partial^c f_i(\overline{x}),\ i \in K, \\ 0 = -g_j(\overline{x}) \geqq \beta_j(x, \overline{x}) \zeta_j \eta(x, \overline{x}), \quad \text{for any } \zeta_j \in \partial^c g_j(\overline{x}),\ j \in J(\overline{x}). \end{gathered} \tag{3.2}$$

By using $\alpha_i(x, \overline{x}) > 0,\ i \in K$ and $\beta_j(x, \overline{x}) > 0,\ j \in J(\overline{x})$, we get

$$\begin{gathered} \frac{1}{\alpha_i(x, \overline{x})} f_i(x) - \frac{1}{\alpha_i(x, \overline{x})} f_i(\overline{x}) \geqq \xi_i \eta(x, \overline{x}), \quad \text{for any } \xi_i \in \partial^c f_i(\overline{x}),\ i \in K, \\ 0 \geqq \zeta_j \eta(x, \overline{x}) \quad \text{for any } \zeta_j \in \partial^c g_j(\overline{x}),\ j \in J(\overline{x}). \end{gathered} \tag{3.3}$$

As $\overline{\lambda}_i > 0,\ i \in K$ and $\overline{\mu}_j \geqq 0,\ j \in J(\overline{x})$, using (3.3), we obtain

$$\sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_i(x, \overline{x})} f_i(x) - \sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_i(x, \overline{x})} f_i(\overline{x}) \geqq \left(\sum_{i=1}^{k} \overline{\lambda}_i \xi_i + \sum_{j \in J(\overline{x})} \overline{\mu}_j \zeta_j \right) \eta(x, \overline{x}), \tag{3.4}$$

which on using (3.1) yields

$$\sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_i(x,\overline{x})} f_i(x) \geqq \sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_i(x,\overline{x})} f_i(\overline{x}). \tag{3.5}$$

Suppose that $\overline{x}$ is not an efficient solution for (MP). Then, there exist a feasible solution x for (MP) and an index r such that

$$\begin{gathered} f_r(x) < f_r(\overline{x}), \\ f_i(x) \leqq f_i(\overline{x}), \quad \forall i \neq r. \end{gathered} \tag{3.6}$$

Because $\overline{\lambda}_i > 0$, and $\alpha_i(x,\overline{x}) > 0, i \in K$, we have

$$\sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_i(x,\overline{x})} f_i(x) < \sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_i(x,\overline{x})} f_i(\overline{x}). \tag{3.7}$$

This contradicts inequality (3.5), and $\overline{x}$ is thus an efficient solution for (MP). □

Theorem 3.2. *Suppose that there exist a feasible solution $\overline{x}$ for* (MP) *and scalars $\overline{\lambda}_i > 0, i \in K, \overline{\mu}_j \geqq 0, j \in J(\overline{x})$ such that*

(i) $0 \in \sum_{i=1}^{k} \overline{\lambda}_i \partial^c f_i(\overline{x}) + \sum_{j\in J(\overline{x})} \overline{\mu}_j \partial^c g_j(\overline{x})$,

(ii) $(f^{\overline{\lambda}}, g_J^{\overline{\mu}})$ *is V-pseudo-quasi-type I invex at $\overline{x}$.*

Then, $\overline{x}$ is an efficient solution for (MP).

Proof. Suppose that $\overline{x}$ is not an efficient solution for (MP). Then, there exist a feasible solution x for (MP) and an index r such that

$$\begin{gathered} f_r(x) < f_r(\overline{x}), \\ f_i(x) \leqq f_i(\overline{x}), \quad \forall i \neq r. \end{gathered} \tag{3.8}$$

Since $\overline{\lambda}_i > 0$ and $\tilde{\alpha}_i(x,\overline{x}) > 0$, $i \in K$, above inequalities give

$$\sum_{i=1}^{k} \overline{\lambda}_i \tilde{\alpha}_i(x,\overline{x}) f_i(x) < \sum_{i=1}^{k} \overline{\lambda}_i \tilde{\alpha}_i(x,\overline{x}) f_i(\overline{x}). \tag{3.9}$$

Also $g_j(\overline{x}) = 0$, $j \in J(\overline{x})$ yields

$$\sum_{j\in J(\overline{x})} \tilde{\beta}_j(x,\overline{x}) \overline{\mu}_j g_j(\overline{x}) = 0. \tag{3.10}$$

The hypothesis (ii) and inequalities (3.9) and (3.10) imply

$$\begin{aligned} &\sum_{i=1}^{k} \xi_i' \eta(x,\overline{x}) < 0, \quad \text{for any } \xi_i' \in \partial^c\left(\overline{\lambda}_i f_i\right)(\overline{x}), \\ &\sum_{j\in J(\overline{x})} \zeta_j' \eta(x,\overline{x}) \leqq 0, \quad \text{for any } \zeta_j' \in \partial^c\left(\overline{\mu}_j g_j\right)(\overline{x}). \end{aligned} \tag{3.11}$$

Adding these inequalities, we obtain

$$\left(\sum_{i=1}^{k} \xi_i' + \sum_{j\in J(\overline{x})} \zeta_j'\right)\eta(x,\overline{x}) < 0, \tag{3.12}$$

but, by Theorem 2.7, for some $\xi_i' \in \partial^c(\overline{\lambda}_i f_i)(\overline{x})$ and $\zeta_j' \in \partial^c(\overline{\mu}_j g_j)(\overline{x})$, there exist $\xi_i \in \partial^c f_i(\overline{x})$ and $\zeta_j \in \partial^c g_j(\overline{x})$ such that

$$\xi_i' = \overline{\lambda}_i \xi_i, \quad i \in K \text{ and } \zeta_j' = \overline{\mu}_j \zeta_j, \quad j \in J(\overline{x}). \tag{3.13}$$

Hence, the above inequality becomes

$$\left(\sum_{i=1}^{k} \overline{\lambda}_i \xi_i + \sum_{j\in J(\overline{x})} \overline{\mu}_j \zeta_j\right)\eta(x,\overline{x}) < 0. \tag{3.14}$$

This contradicts (i), as for $\xi_i \in \partial^c f_i(\overline{x}), \zeta_j \in \partial^c g_j(\overline{x}), \sum_{i=1}^{k} \overline{\lambda}_i \xi_i + \sum_{j\in J(\overline{x})} \overline{\mu}_j \zeta_j = 0$. Hence, $\overline{x}$ is an efficient solution for (MP). □

Remark 3.3. If we take $\overline{\lambda}_i \geqq 0$, $i \in K$, $\sum_{i=1}^{k} \overline{\lambda}_i = 1$, then the above theorem still holds under the assumption that $(f^{\overline{\lambda}}, g_J^{\overline{\mu}})$ is V-strictly pseudo-quasi-type I invex at $\overline{x}$.

Theorem 3.4. *Suppose that there exist a feasible solution $\overline{x}$ for* (MP) *and scalars $\overline{\lambda}_i \geqq 0$, $i \in K$, $\sum_{i=1}^{k} \overline{\lambda}_i = 1, \overline{\mu}_j \geqq 0, j \in J(\overline{x})$ such that*

(i) $0 \in \sum_{i=1}^{k} \overline{\lambda}_i \partial^c f_i(\overline{x}) + \sum_{j\in J(\overline{x})} \overline{\mu}_j \partial^c g_j(\overline{x})$,

(ii) (f, g_J) *is V-type I invex at* $\overline{x}$.

Then, $\overline{x}$ is a weakly efficient solution for (MP).

Proof. Following the proof of Theorem 3.1, we obtain

$$\sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_i(x,\overline{x})} f_i(x) \geqq \sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_i(x,\overline{x})} f_i(\overline{x}). \tag{3.15}$$

Suppose that $\overline{x}$ is not a weakly efficient solution for (MP). Then, there exists a feasible solution $x(x \neq \overline{x})$ for (MP) such that

$$f_i(x) < f_i(\overline{x}), \quad i \in K. \tag{3.16}$$

Because $\overline{\lambda}_i \geqq 0, i \in K, \sum_{i=1}^{k} \overline{\lambda}_i = 1$, and $\alpha_i(x, \overline{x}) > 0, i \in K$, we have

$$\sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_i(x, \overline{x})} f_i(x) < \sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_i(x, \overline{x})} f_i(\overline{x}). \tag{3.17}$$

This contradicts inequality (3.15), and $\overline{x}$ is thus a weakly efficient solution for (MP). □

Theorem 3.5. *Suppose that there exist a feasible solution $\overline{x}$ for* (MP) *and scalars $\overline{\lambda}_i \geqq 0, i \in K, \sum_{i=1}^{k} \overline{\lambda}_i = 1$ and $\overline{\mu}_j \geqq 0, j \in J(\overline{x})$ such that*

(i) $0 \in \sum_{i=1}^{k} \overline{\lambda}_i \partial^c f_i(\overline{x}) + \sum_{j \in J(\overline{x})} \overline{\mu}_j \partial^c g_j(\overline{x})$,

(ii) $(f^{\overline{\lambda}}, g_J^{\overline{\mu}})$ *is V-pseudo-quasi-type I invex at $\overline{x}$.*

Then, $\overline{x}$ is a weakly efficient solution for (MP)*.*

Proof. Suppose that $\overline{x}$ is not a weakly efficient solution for (MP). Then, there exists a feasible solution x $(x \neq \overline{x})$ for (MP) such that

$$f_i(x) < f_i(\overline{x}), \quad i \in K. \tag{3.18}$$

Since $\overline{\lambda}_i \geqq 0, i \in K, \sum_{i=1}^{k} \overline{\lambda}_i = 1$, and $\tilde{\alpha}_i(x, \overline{x}) > 0, i \in K$, the above inequality gives

$$\sum_{i=1}^{k} \overline{\lambda}_i \tilde{\alpha}_i(x, \overline{x}) f_i(x) < \sum_{i=1}^{k} \overline{\lambda}_i \tilde{\alpha}_i(x, \overline{x}) f_i(\overline{x}). \tag{3.19}$$

The remaining part of the proof is similar to that of Theorem 3.2. □

Theorem 3.6. *Suppose that there exist a feasible solution $\overline{x}$ for* (MP) *and scalars $\overline{\lambda}_i \geqq 0, i \in K, \sum_{i=1}^{k} \overline{\lambda}_i = 1$ and $\overline{\mu}_j \geqq 0, j \in J(\overline{x})$ such that*

(i) $0 \in \sum_{i=1}^{k} \overline{\lambda}_i \partial^c f_i(\overline{x}) + \sum_{j \in J(\overline{x})} \overline{\mu}_j \partial^c g_j(\overline{x})$,

(ii) $(f^{\overline{\lambda}}, g_J^{\overline{\mu}})$ *is V-quasistrictly pseudo-type I invex at $\overline{x}$.*

Then, $\overline{x}$ is a weakly efficient solution for (MP)*.*

Proof. Suppose that $\overline{x}$ is not a weakly efficient solution for (MP). Then, there exists a feasible solution x $(x \neq \overline{x})$ for (MP) such that

$$f_i(x) < f_i(\overline{x}), \quad i \in K. \tag{3.20}$$

Since $\overline{\lambda}_i \geqq 0, i \in K, \sum_{i=1}^{k} \overline{\lambda}_i = 1$, and $\widehat{\alpha}_i(x,\overline{x}) > 0, i \in K$, the above inequality gives

$$\sum_{i=1}^{k} \overline{\lambda}_i \widehat{\alpha}_i(x,\overline{x}) f_i(x) < \sum_{i=1}^{k} \overline{\lambda}_i \widehat{\alpha}_i(x,\overline{x}) f_i(\overline{x}). \tag{3.21}$$

Also $g_j(\overline{x}) = 0, j \in J(\overline{x})$ yields

$$\sum_{j\in J(\overline{x})} \widehat{\beta}_j(x,\overline{x}) \overline{\mu}_j g_j(\overline{x}) = 0. \tag{3.22}$$

If hypothesis (ii) holds, we have

$$\sum_{i=1}^{k} \xi_i' \eta(x,\overline{x}) > 0 \Longrightarrow \sum_{i=1}^{k} \overline{\lambda}_i \widehat{\alpha}_i(x,\overline{x}) f_i(x) > \sum_{i=1}^{k} \overline{\lambda}_i \widehat{\alpha}_i(x,\overline{x}) f_i(\overline{x}), \quad \text{for any } \xi_i' \in \partial^c\left(\overline{\lambda}_i f_i\right)(\overline{x}), \tag{3.23}$$

$$-\sum_{j\in J(\overline{x})} \widehat{\beta}_j(x,\overline{x}) \overline{\mu}_j g_j(\overline{x}) \leqq 0 \Longrightarrow \sum_{j\in J(\overline{x})} \zeta_j' \eta(x,\overline{x}) < 0, \quad \text{for any } \zeta_j' \in \partial^c\left(\overline{\mu}_j g_j\right)(\overline{x}). \tag{3.24}$$

In view of (3.22), (3.24) implies

$$\sum_{j\in J(\overline{x})} \zeta_j' \eta(x,\overline{x}) < 0, \quad \text{for any } \zeta_j' \in \partial^c\left(\overline{\mu}_j g_j\right)(\overline{x}). \tag{3.25}$$

Also, by assumption (i) and Theorem 2.7, we have

$$\sum_{i=1}^{k} \xi_i' + \sum_{j\in J(\overline{x})} \zeta_j' = 0. \tag{3.26}$$

Therefore, (3.25) becomes

$$\sum_{i=1}^{k} \xi_i' \eta(x,\overline{x}) > 0, \quad \text{for any } \xi_i' \in \partial^c\left(\overline{\lambda}_i f_i\right)(\overline{x}). \tag{3.27}$$

In view of (3.27), (3.23) yields

$$\sum_{i=1}^{k} \overline{\lambda}_i \widehat{\alpha}_i(x,\overline{x}) f_i(x) > \sum_{i=1}^{k} \overline{\lambda}_i \widehat{\alpha}_i(x,\overline{x}) f_i(\overline{x}), \tag{3.28}$$

which contradicts (3.21). Hence, $\overline{x}$ is a weakly efficient solution for (MP). □

4. Mond-Weir Type Duality

We now consider the following Mond-Weir type dual for (MP):

$$\text{(MWD) Maximize} \quad f(y) \qquad \text{subject to} \tag{4.1}$$

$$0 \in \sum_{i=1}^{k} \lambda_i \partial^c f_i(y) + \sum_{j=1}^{m} \mu_j \partial^c g_j(y), \tag{4.2}$$

$$\mu_j g_j(y) \geqq 0, \quad j \in M, \tag{4.3}$$

$$\lambda_i \geqq 0, \quad i \in K, \tag{4.4}$$

$$\mu_j \geqq 0, \quad j \in M, \tag{4.5}$$

$$\sum_{i=1}^{k} \lambda_i = 1. \tag{4.6}$$

Let T be the set of all feasible solutions of (MWD).

Theorem 4.1 (weak duality). *Let $x \in S$ and $(y, \lambda, \mu) \in T$ such that (f^λ, g^μ) is V-pseudo-quasi-type I invex at y. Then the following cannot hold*

$$f(x) < f(y). \tag{4.7}$$

Proof. Suppose the contrary to the result that (4.7) holds, that is, $f(x) < f(y)$. Using $\tilde{\alpha}_i(x, y) > 0$, $\lambda_i \geqq 0$, $i \in K$ and $\sum_{i=1}^{k} \lambda_i = 1$, we get

$$\sum_{i=1}^{k} \tilde{\alpha}_i(x, y) \lambda_i f_i(x) < \sum_{i=1}^{k} \tilde{\alpha}_i(x, y) \lambda_i f_i(y). \tag{4.8}$$

Also, as $\tilde{\beta}_j(x, y) > 0$, $j \in M$, inequality (4.3) yields

$$-\sum_{j=1}^{m} \tilde{\beta}_j(x, y) \mu_j g_j(y) \leqq 0. \tag{4.9}$$

By V-pseudo-quasi-type I invexity of (f^λ, g^μ) at y, (4.8) and (4.9) give

$$\begin{gathered} \sum_{i=1}^{k} \xi_i' \eta(x, y) < 0, \quad \text{for any } \xi_i' \in \partial^c(\lambda_i f_i)(y), \\ \sum_{j=1}^{m} \zeta_j' \eta(x, y) \leqq 0, \quad \text{for any } \zeta_j' \in \partial^c(\mu_j g_j)(y). \end{gathered} \tag{4.10}$$

Adding the above inequalities, we get

$$\left(\sum_{i=1}^{k}\xi_i' + \sum_{j=1}^{m}\zeta_j'\right)\eta(x,y) < 0. \tag{4.11}$$

However, by Theorem 2.7, for some $\xi_i' \in \partial^c(\lambda_i f_i)(y)$ and $\zeta_j' \in \partial^c(\mu_j g_j)(y)$, there exist $\xi_i \in \partial^c f_i(y)$ and $\zeta_j \in \partial^c g_j(y)$ such that

$$\xi_i' = \lambda_i\xi_i, \quad i \in K \text{ and } \zeta_j' = \mu_j\zeta_j, \quad j \in M. \tag{4.12}$$

Hence, the above inequality changes to

$$\left(\sum_{i=1}^{k}\lambda_i\xi_i + \sum_{j=1}^{m}\mu_j\zeta_j\right)\eta(x,y) < 0, \tag{4.13}$$

which contradicts the dual constraint (4.2), because $\xi_i \in \partial^c f_i(y)$ and $\zeta_j \in \partial^c g_j(y)$ imply $\sum_{i=1}^{k}\lambda_i\xi_i + \sum_{j=1}^{m}\mu_j\zeta_j = 0$. Hence, (4.7) cannot hold. □

Definition 4.2 (Cottle's constraint qualification [21, page 48]). Let $f_i, i \in K$ and $g_j, j \in M$ be locally Lipschitz functions at a point $u \in X$. The problem (MP) is said to satisfy Cottle's constraint qualification at u if either $g_j(u) < 0$ for all $j \in M$ or $0 \in \operatorname{conv}\{\partial^c g_j(u) : g_j(u) = 0\}$, where $\operatorname{conv} Z$ denotes the convex hull of the set Z.

Theorem 4.3 (Karush-Kuhn-Tucker type necessary conditions [21, page 50]). *Assume that $\overline{x}$ is a weakly efficient solution for* (MP) *at which Cottle's constraint qualification is satisfied. Then, there exist scalars $\overline{\lambda}_i \geqq 0, i \in K, \sum_{i=1}^{k}\overline{\lambda}_i = 1$ and $\overline{\mu}_j \geqq 0, j \in M$ such that*

$$\begin{gathered} 0 \in \sum_{i=1}^{k}\overline{\lambda}_i\partial^c f_i(\overline{x}) + \sum_{j=1}^{m}\overline{\mu}_j\partial^c g_j(\overline{x}), \\ \overline{\mu}_j g_j(\overline{x}) = 0, \quad j \in M. \end{gathered} \tag{4.14}$$

Theorem 4.4 (strong duality). *Let $\overline{x}$ be a weakly efficient solution for* (MP) *at which Cottle's constraint qualification is satisfied. Then, there exist $\overline{\lambda} \in \mathbb{R}^k$, $\overline{\mu} \in \mathbb{R}^m$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is feasible for (MWD) and the objective values of* (MP) *and (MWD) are equal. Further, if the hypotheses of weak duality (Theorem 4.1) hold for all feasible solutions (y, λ, μ) for (MWD), then $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a weakly efficient solution of (MWD).*

Proof. Since $\overline{x}$ is a weakly efficient solution of (MP) and the Cottle's constraint qualification is satisfied at $\overline{x}$, from Theorem 4.3, there exist $\overline{\lambda}_i \geqq 0$, $i \in K$, $\sum_{i=1}^{k} \overline{\lambda}_i = 1$, and $\overline{\mu}_j \geqq 0$, $j \in M$ such that

$$0 \in \sum_{i=1}^{k} \overline{\lambda}_i \partial^c f_i(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \partial^c g_j(\overline{x}),$$
$$\overline{\mu}_j g_j(\overline{x}) = 0, \quad j \in M, \tag{4.15}$$

which yields that $(\overline{x}, \overline{\lambda}, \overline{\mu})$is feasible for (MWD) and the corresponding objective values are equal. If $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is not a weakly efficient solution for (MWD), then there exists a feasible solution (y, λ, μ) for (MWD) such that

$$f(\overline{x}) < f(y), \tag{4.16}$$

which contradicts the weak duality (Theorem 4.1). Hence, $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a weakly efficient solution for (MWD). □

Theorem 4.5 (strict converse duality). *Let $\overline{x} \in S$ and $(\overline{y}, \overline{\lambda}, \overline{\mu}) \in T$ such that*

$$\sum_{i=1}^{k} \overline{\lambda}_i f_i(\overline{x}) \leqq \sum_{i=1}^{k} \overline{\lambda}_i f_i(\overline{y}). \tag{4.17}$$

If

(i) $(f^{\overline{\lambda}}, g^{\overline{\mu}})$ *is V-strictly pseudo-quasi-type I invex at* $\overline{y}$,

(ii) $\tilde{\alpha}_i^1(\overline{x}, \overline{y}) = 1$, $i \in K$,

then $\overline{x} = \overline{y}$.

Proof. We assume that $\overline{x} \neq \overline{y}$ and exhibit a contradiction. Since $(\overline{y}, \overline{\lambda}, \overline{\mu}) \in T$, from (4.2), there exist $\overline{\xi}_i \in \partial^c f_i(\overline{y})$, $i \in K$ and $\overline{\zeta}_j \in \partial^c g_j(\overline{y})$, $j \in M$ such that

$$\sum_{i=1}^{k} \overline{\lambda}_i \overline{\xi}_i + \sum_{j=1}^{m} \overline{\mu}_j \overline{\zeta}_j = 0. \tag{4.18}$$

The hypothesis (i) along with (4.3) and $\tilde{\beta}_j(\overline{x}, \overline{y}) > 0$, $j \in M$ yields

$$\sum_{j=1}^{m} \overline{\zeta'_j} \eta(\overline{x}, \overline{y}) \leqq 0, \quad \text{for any } \overline{\zeta'_j} \in \partial^c \left(\overline{\mu}_j g_j\right)(\overline{y}). \tag{4.19}$$

Also by Theorem 2.7, for some $\overline{\xi}_i \in \partial^c f_i(\overline{y})$, $i \in K$ and $\overline{\zeta}_j \in \partial^c g_j(\overline{y})$, $j \in M$, there exist $\overline{\xi'_i} \in \partial^c(\overline{\lambda}_i f_i)(\overline{y})$ and $\overline{\zeta'_j} \in \partial^c(\overline{\mu}_j g_j)(\overline{y})$ such that $\overline{\xi'_i} = \overline{\lambda}_i \overline{\xi}_i$ and $\overline{\zeta'_j} = \overline{\mu}_j \overline{\zeta}_j$.

Hence, (4.18) gives

$$\left(\sum_{i=1}^{k}\overline{\xi_i'}+\sum_{j=1}^{m}\overline{\zeta_j'}\right)\eta(\overline{x},\overline{y})=0, \tag{4.20}$$

which with (4.19) gives

$$\sum_{i=1}^{k}\overline{\xi_i'}\eta(\overline{x},\overline{y})\geqq 0, \quad \text{for any } \overline{\xi_i'}\in\partial^c\left(\overline{\lambda}_i f_i\right)(\overline{y}). \tag{4.21}$$

Therefore the hypothesis (i) again yields

$$\sum_{i=1}^{k}\widetilde{\alpha}_i(\overline{x},\overline{y})\overline{\lambda}_i f_i(\overline{x})>\sum_{i=1}^{k}\widetilde{\alpha}_i(\overline{x},\overline{y})\overline{\lambda}_i f_i(\overline{y}). \tag{4.22}$$

Since $\widetilde{\alpha}_i(\overline{x},\overline{y})=1, i\in K$, we have

$$\sum_{i=1}^{k}\overline{\lambda}_i f_i(\overline{x})>\sum_{i=1}^{k}\overline{\lambda}_i f_i(\overline{y}), \tag{4.23}$$

which contradicts (4.17). This completes the proof. □

References

[1] M. A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 2, pp. 545–550, 1981.

[2] B. D. Craven, "Invex functions and constrained local minima," *Bulletin of the Australian Mathematical Society*, vol. 24, no. 3, pp. 357–366, 1981.

[3] M. A. Hanson and B. Mond, "Necessary and sufficient conditions in constrained optimization," *Mathematical Programming*, vol. 37, no. 1, pp. 51–58, 1987.

[4] N. G. Rueda and M. A. Hanson, "Optimality criteria in mathematical programming involving generalized invexity," *Journal of Mathematical Analysis and Applications*, vol. 130, no. 2, pp. 375–385, 1988.

[5] R. N. Kaul, S. K. Suneja, and M. K. Srivastava, "Optimality criteria and duality in multiple-objective optimization involving generalized invexity," *Journal of Optimization Theory and Applications*, vol. 80, no. 3, pp. 465–482, 1994.

[6] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley-Interscience, New York, NY, USA, 1983.

[7] B. D. Craven, "Nondifferentiable optimization by smooth approximations," *Optimization*, vol. 17, no. 1, pp. 3–17, 1986.

[8] T. W. Reiland, "Nonsmooth invexity," *Bulletin of the Australian Mathematical Society*, vol. 42, no. 3, pp. 437–446, 1990.

[9] R. R. Egudo and M. A. Hanson, "On sufficiency of the Kuhn-Tucker conditions in nonsmooth multiobjective programming," Tech. Rep. M-888, Florida State University, 1993.

[10] G. Giorgi and A. Guerraggio, "The notion of invexity in vector optimization: smooth and nonsmooth case," in *Generalized Convexity, Generalized Monotonicity: Recent Results*, J. P. Crouzeix, J. E. Martinez-Legaz, and M. Volle, Eds., vol. 27, pp. 389–405, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.

[11] M. H. Kim and G. M. Lee, “On duality theorems for nonsmooth Lipschitz optimization problems,” *Journal of Optimization Theory and Applications*, vol. 110, no. 3, pp. 669–675, 2001.

[12] D. S. Kim and S. Schaible, “Optimality and duality for invex nonsmooth multiobjective programming problems,” *Optimization*, vol. 53, no. 2, pp. 165–176, 2004.

[13] D. S. Kim and H. J. Lee, “Optimality conditions and duality in nonsmooth multiobjective programs,” *Journal of Inequalities and Applications*, Article ID 939537, 12 pages, 2010.

[14] H. Kuk, G. M. Lee, and D. S. Kim, “Nonsmooth multiobjective programs with $V-\rho-$invexity,” *Indian Journal of Pure and Applied Mathematics*, vol. 29, no. 4, pp. 405–412, 1998.

[15] H. Kuk and T. Tanino, “Optimality and duality in nonsmooth multiobjective optimization involving generalized type I functions,” *Computers & Mathematics with Applications*, vol. 45, no. 10-11, pp. 1497–1506, 2003.

[16] G. M. Lee, “Nonsmooth invexity in multiobjective programming,” *Journal of Information & Optimization Sciences*, vol. 15, no. 1, pp. 127–136, 1994.

[17] S. K. Mishra and R. N. Mukherjee, “On generalised convex multi-objective nonsmooth programming,” *Australian Mathematical Society B*, vol. 38, no. 1, pp. 140–148, 1996.

[18] V. Jeyakumar and B. Mond, “On generalised convex mathematical programming,” *Australian Mathematical Society B*, vol. 34, no. 1, pp. 43–53, 1992.

[19] F. A. Zhao, “On sufficiency of the Kuhn-Tucker conditions in nondifferentiable programming,” *Bulletin of the Australian Mathematical Society*, vol. 46, no. 3, pp. 385–389, 1992.

[20] M. A. Hanson, R. Pini, and C. Singh, “Multiobjective programming under generalized type I invexity,” *Journal of Mathematical Analysis and Applications*, vol. 261, no. 2, pp. 562–577, 2001.

[21] K. M. Miettinen, *Nonlinear Multiobjective Optimization*, Kluwer Academic Publishers, Boston, Mass, USA, 1999.